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PROGRAM DESIGN OF A DIFFERENTIAL GAME WITH INTEGRAL PAYOFF*

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The functional form of the method of auxiliary program constructions is justified for a position differential game with an integral payoff depending explicitly on the realizations of the object's motion and of the controls. The paper is closely related to /1–4/.

1. We consider the problem of position controls u and v which minimize-maximize a prescribed functional

$$\gamma(x(t_*[\cdot]\vartheta), u(t_*[\cdot]\vartheta), v(t_*[\cdot]\vartheta)) = \int_{t_*}^{\vartheta} \omega(t, x[t], u[t], v[t])\,dt + \varphi(x[\vartheta]) \tag{1.1}$$

on the motions $x(t_*[\cdot]\vartheta) = \{x[t],\ t_* \leqslant t \leqslant \vartheta\}$ of the object

$$\begin{gathered} x^{\cdot} = f(t, x, u, v),\ t_0 \leqslant t \leqslant \vartheta,\ u \in P,\quad v \in Q \\ \| f(t, x, u, v) \| \leqslant \chi(1 + \| x \|),\ \chi = \text{const} \end{gathered} \tag{1.2}$$

and on the control realizations

$$u(t_*[\cdot]\vartheta) = \{u[t] \in P,\ t_* \leqslant t \leqslant \vartheta\},\qquad v(t_*[\cdot]\vartheta) = \{v[t] \in Q,\ t_* \leqslant t \leqslant \vartheta\}.$$

Here $x \in R^n$, $\|x\|$ is the Euclidean norm of vector x, t_0 and ϑ are fixed, $t_* \in [t_0, \vartheta)$, $P \subset R^p$, $Q \subset R^q$ are compacta, the functions f and ω are continuous in t, x, u, v and the functions f, ω, φ satisfy a Lipschitz condition in x in every bounded domain G. In addition, the condition (/1/, p.97)

$$\begin{gathered} \min_{u\in P}\max_{v\in Q} \{\langle s\cdot f(t, x, u, v)\rangle + s_{n+1}\cdot\omega(t, x, u, v)\} = \\ \max_{v\in Q}\min_{u\in P} \{\langle s\cdot f(t, x, u, v)\rangle + s_{n+1}\cdot\omega(t, x, u, v)\} \end{gathered}$$

is fulfilled. Here s is an n-dimensional vector, s_{n+1} is a scalar, $\langle s\cdot f\rangle$ is a scalar product. The admissible control laws, i.e., the strategies $u(\cdot)$ and $v(\cdot)$, are identified with the functions

$$\begin{gathered} u(\cdot) = \{u(t, x, \varepsilon) \in P,\ t_0 \leqslant t \leqslant \vartheta,\ x \in R^n,\ \varepsilon > 0\} \\ v(\cdot) = \{v(t, x, \varepsilon) \in Q,\ t_0 \leqslant t \leqslant \vartheta,\ x \in R^n,\ \varepsilon > 0\} \end{gathered}$$

($\{t, x\}$ is the position, ε is some parameter). For a given initial position $\{t_*, x_*\}$, for chosen $\varepsilon > 0$ and partition $\Delta = \{\tau_i\}$, $\tau_0 = t_*$, $\tau_{m+1} = \vartheta$, and for some t-measurable control realization $v(t_*[\cdot]\vartheta)$ the chosen strategy $u(\cdot)$ generates a motion $x(t_*[\cdot]\vartheta)$ of object (1.2) by steps, as an absolutely continuous solution of the equation

$$\begin{gathered} x^{\cdot}[t] = f(t, x[t], u(\tau_i, x[\tau_i], \varepsilon), v[t]) \\ x[t_*] = x_*,\quad \tau_i \leqslant t \leqslant \tau_{i+1},\quad i = 0, \ldots, m \end{gathered} \tag{1.3}$$

The motion $x(t_*[\cdot]\vartheta)$ of objects (1.2), generated from the position $\{t_*, x_*\}$ by the strategy $v(\cdot)$, is defined analogously as the absolutely continuous solution $x[t]$ of the stepwise differential equation

$$\begin{gathered} x^{\cdot}[t] = f(t, x[t], u[t], v(\tau_i, x[\tau_i], \varepsilon)) \\ x[t_*] = x_*,\quad \tau_i \leqslant t \leqslant \tau_{i+1},\quad i = 0, \ldots, m \end{gathered} \tag{1.4}$$

We allow initial states $x[t_*] = x_*$ from a domain $G[t_*] = \{\|x\| \leqslant (r_0 + 1)\exp[\chi\cdot(t_* - t_0)] - 1\}$, $t_0 \leqslant t_* \leqslant \vartheta$, where r_0 is a preselected number. When $x_* \in G[t_*]$ any motions $x(t_*[\cdot]t^*)$, $t^* \leqslant \vartheta$ of object (1.2) do not leave the domain $G_* = \{\{t, x\}: x \in G[t],\ t_* \leqslant t \leqslant t^*\}$. The following statement is valid.

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Statement. The differential game being examined has a value $\rho(t, x)$ and a saddle point $\{u^\circ(\cdot), v^\circ(\cdot)\}$ uniform with respect to the initial position $\{t, x\}$ from domain $G_0 = \{\{t, x\}: x \in G[t], t_0 \leqslant t \leqslant \vartheta\}$.

This signifies that for any $\zeta > 0$ we can find $\varepsilon_*(\zeta) > 0$ and $\delta(\varepsilon, \zeta) > 0$ such that for every motion $x^\circ(t_*[\cdot]\vartheta)$, $x^\circ[t_*] = x_*$ generated by strategy $u^\circ(\cdot)$ in accord with scheme (1.3) the inequality

$$\gamma(x^\circ(t_*[\cdot]\vartheta), u^\circ(t_*[\cdot]\vartheta), v(t_*[\cdot]\vartheta)) \leqslant \rho(t_*, x_*) + \zeta \tag{1.5}$$

is fulfilled for any initial position $\{t_*, x_*\} \in G_0$, which for every motion $x^\circ(t_*[\cdot]\vartheta)$, $x^\circ[t_*] = x_*$ generated by strategy $v^\circ(\cdot)$ in accord with scheme (1.4) the inequality

$$\gamma(x^\circ(t_*[\cdot]\vartheta), u(t_*[\cdot]\vartheta), v^\circ(t_*[\cdot]\vartheta)) \geqslant \rho(t_*, x_*) - \zeta \tag{1.6}$$

is fulfilled if only the conditions

$$\varepsilon \leqslant \varepsilon_*(\zeta), \ \max_i [\tau_{i+1} - \tau_i] \leqslant \delta(\varepsilon, \zeta) \tag{1.7}$$

are fulfilled. In (1.5) and (1.6) $u^\circ(t_*[\cdot]\vartheta)$ and $v^\circ(t_*[\cdot]\vartheta)$ are realizations of strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$, computed along the motions $x^\circ(t_*[\cdot]\vartheta)$ generated by them. From (1.5) and (1.6) it follows, in particular, that on the motions $x^{\circ\circ}(t_*[\cdot]\vartheta]$ generated simultaneously (possibly for different Δ) by the strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$ and on the corresponding realizations $u^\circ(t_*[\cdot]\vartheta)$ and $v^\circ(t_*[\cdot]\vartheta)$ the value of functional (1.1) differs arbitrarily little from the game's value $\rho(t_*, x_*)$ if only conditions (1.7) are fulfilled. We omit the proof of this assertion. With necessary changes it can be carried out by the scheme in /2,3/. The optimal strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$ are constructed reasonably effectively from the known game value $\rho(t, x)$ by the same plan as in /2/ (p.426 of English translation).

2. The aim of the present paper is to justify an auxiliary program construction which under certain assumptions permits the determination of the value of the differential game at hand in the case when the object's Eq.(1.2) has the form

$$x^\cdot = A(t)x + f(t, u, v) \tag{2.1}$$

Here $A(t)$ is a matrix-valued function, while the function ω in (1.1) has the form

$$\omega(t, x, u, v) = \omega_x(t, x) + \omega_{uv}(t, u, v) \tag{2.2}$$

where the function ω_x is convex in x. In addition, we now assume that the function φ in (1.1) too is convex in x. We consider the model

$$\begin{aligned} w^\cdot &= A(t)w + f(t, u, v), \quad u \in P, \ v \in Q \\ w_{n+1}^\cdot &= \omega_{uv}(t, u, v) \end{aligned} \tag{2.3}$$

Denoting $\{w_1, \ldots, w_n, w_{n+1}\} = z \in R^{n+1}$, we write Eqs.(2.3) as

$$z^\cdot = A_0(t)z + f_0(t, u, v), \quad u \in P, \quad v \in Q \tag{2.4}$$

where $A_0(t)$ is an $(n+1) \times (n+1)$-matrix and $f_0(t, u, v)$ is the vector $f(t, u, v)$ augmented by the $(n+1)$ st component of $\omega_{uv}(t, u, v)$. On the motions of model (2.4) we consider the functional

$$\gamma_0(z(t_*[\cdot]\vartheta)) = \int_{t_*}^{\vartheta} \omega_0(t, z[t])\,dt + \varphi_0(z[\vartheta]) \tag{2.5}$$

corresponding to functional (1.1) with due regard to (2.2). Here $\omega_0(t, z) = \omega_x(t, w)$, $\varphi_0(z) = \varphi(w) + w_{n+1}$, and both these functions are convex in z.

The essence of the result being discussed is as follows. For the initial game with functional γ as given by (1.1) and (2.2) we can establish, using the model (2.4) with functional γ_0 as given by (2.5), including the additional coordinate w_{n+1}, the possibility of applying the method of auxiliary program constructions such that the value and the strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$ could be constructed on the basis of information only on the current position $\{t, x[t]\}$. When passing from the model to the object the additional coordinate w_{n+1} is eliminated because of the approach developed for position functionals /3/. Here the intermediate program construction is of a functional nature. We proceed with the description of its elements.

Any measurable function $v(t^*[\cdot]\vartheta) = \{v[\tau] \in Q, t^* \leqslant \tau \leqslant \vartheta\}$, is called an action $v(t^*[\cdot]\vartheta)$, $t_* \leqslant t^* < \vartheta$. We take certain $v^* \in Q$, $\tau^* \in [t^*, \vartheta]$ and we consider the set

$$F(\tau^*, v^*) = \overline{\mathrm{co}}\,\{f_0(\tau^*, u, v^*), u \in P\} \tag{2.6}$$

bounded, closed and convex in R^{n+1}. Let $L^{(2)}[t^*, \vartheta]$ be the space of measurable functions $\{g(t^*[\cdot]\vartheta)\}$ with the norm

$$\|g(t^*[\cdot]\vartheta)\|_2 = \left(\int_{t_*}^{\vartheta} \|g[\tau]\|^2 d\tau\right)^{1/2}$$

Suppose that a certain action $v^*(t^*[\cdot]\vartheta)$ has been chosen. By the symbol $S(v^*(t^*[\cdot]\vartheta))$ we denote the set $\{g(t^*[\cdot]\vartheta)\}$ of elements from $L^{(2)}[t^*, \vartheta]$, satisfying the condition $g[\tau] \in F(\tau, v^*[\tau])$ for almost all $\tau \in [t^*, \vartheta]$. Any function from the set $S(v^*(t^*[\cdot]\vartheta))$ is called an action $g^*(t^*[\cdot]\vartheta)$ consistent with action $v^*(t^*[\cdot]\vartheta)$. We note that this set is nonempty, bounded, convex strongly closed and weakly compact in space $L^{(2)}[t^*, \vartheta]$. The last two properties are proved with the aid of Luzin's theorem /5/, p.291) and the theorem on the separation of sets (/6/, p.452). For a given action $v^*(t^*[\cdot]\vartheta)$, to each fixed action $g^*(t^*[\cdot]\vartheta)$ consistent with it when $z[t^*] = z^*$ there corresponds a motion $z(t^*[\cdot]\vartheta)$ which is represented by the Cauchy formula

$$z[t] = X(t, t^*) z^* + \int_{t_*}^{t} X(t, \tau) g^*[\tau] d\tau, \quad t^* \leqslant t \leqslant \vartheta \tag{2.7}$$

$$z^* \in G^*[t^*] = \{\|w\| \leqslant (r_0 + 2\varepsilon^\circ + 1) \exp[\chi(t^* - t_0)] - 1, \ |w_{n+1}| \leqslant d(t^* - t_0) + 2\varepsilon^\circ\}$$

$$d = \max\{|\omega_{uv}(t, u, v)|, \ t_0 \leqslant t \leqslant \vartheta, \ u \in P, \ v \in Q\}$$

Here $X(t, \tau)$ is the fundamental matrix of solutions of the homogeneous equation corresponding to Eq.(2.4) and $\varepsilon^\circ > 0$ is some fixed number.

Let us consider the space $L_*^{(2)}[t_*, \vartheta]$ of measurable functions $\{z(t_*[\cdot]\vartheta)\}$ with the norm

$$|z(t_*[\cdot]\vartheta)|_2 = \left(\int_{t_*}^{\vartheta} \|z[t]\|^2 dt\right)^{1/2} + |z[\vartheta]| \tag{2.8}$$

where $|z|$ is any chosen norm in space R^{n+1}. We fix a certain function $z^*(t_*[\cdot]t^*) = \{z^*[t], t_* \leqslant t \leqslant t^*\}$, satisfying a Lipschitz condition in t and a certain action $v^*(t^*[\cdot]\vartheta)$. By $W^{(1)}(z^*(t_*[\cdot]t^*), v^*(t^*[\cdot]\vartheta))$ we denote the set of all functions $\{z^{(1)}(t_*[\cdot]\vartheta)\}$ from space $L_*^{(2)}[t_*, \vartheta]$, each of which is pasted together continuously from the chosen function $z^*(t_*[\cdot]t^*)$ and some motion $z(t^*[\cdot]\vartheta)$ from (2.7), generated for action $v^*(t^*[\cdot]\vartheta)$ by any action $g(t^*[\cdot]\vartheta)$ consistent with it when $z^* = z^*[t^*]$. Using the above-mentioned properties of set $S(v^*(t^*[\cdot]\vartheta))$, it can be shown that the set $W^{(1)}(z^*(t_*[\cdot]t^*), v^*(t^*[\cdot]\vartheta))$ is convex and compact in $L_*^{(2)}[t_*, \vartheta]$.

By $W^{(2)}(\beta, M)$ we denote the set of functions $\{z^{(2)}(t_*[\cdot]\vartheta)\}$ from $L_*^{(2)}[t_*, \vartheta]$, such that $\gamma_0(z^{(2)}(t_*[\cdot]\vartheta)) \leqslant \beta$ and $|z^{(2)}(t_*[\cdot]\vartheta)|_2 \leqslant M$, where the functional γ_0 is prescribed by expression (2.5), β any preselected number, M is some sufficiently large fixed number. From what follows we see that we can restrict ourselves to the cases when the set $W^{(2)}(\beta, M)$ is nonempty. From the properties of the functions ω_0 and φ_0 occurring in (2.5) it follows that set $W^{(2)}(\beta, M)$ is bounded, closed and convex in $L_*^{(2)}[t_*, \vartheta]$.

We introduce the space $L_\mu^{(2)}[t_*, \vartheta]$ of measurable functions with scalar product

$$(l(t_*[\cdot]\vartheta) \cdot z(t_*[\cdot]\vartheta)) = \int_{t_*}^{\vartheta} \langle l[t] \cdot z[t]\rangle \mu(dt) = \int_{t_*}^{\vartheta} \langle l[t] \cdot z[t]\rangle dt + \langle l[\vartheta] \cdot z[\vartheta]\rangle$$

From the assumptions made and the properties ensuing from them of the sets $W^{(1)} = W^{(1)}(z^*(t_*[\cdot]t^*), v^*(t^*[\cdot]\vartheta))$ and $W^{(2)} = W^{(2)}(\beta, M)$ it follows that for values of β for which these sets do not intersect, they can be separated (/6/, p.452), while the distance in space $L_*^{(2)}[t_*, \vartheta]$ maximal with respect to all possible actions $v(t^*[\cdot]\vartheta)$, between the sets $W^{(1)}$ and the set $W^{(2)}$ is characterized by the quantity

$$\sigma(z^*(t_*[\cdot]t^*)\beta, M) = \max_{|l(t_*[\cdot]\vartheta)|_2^* \leqslant 1} \varkappa(z^*(t_*[\cdot]t^*), \beta, M, l(t_*[\cdot]\vartheta)) \tag{2.9}$$

$$\varkappa(z^*(t_*[\cdot]t^*), \beta, M, l(t_*[\cdot]\vartheta)) = \max_{v(t^*[\cdot]\vartheta)} \min_{g(t^*[\cdot]\vartheta)} (l(t_*[\cdot]\vartheta) \cdot z^{(1)}(t_*[\cdot]\vartheta)) - \max_{z^{(2)}(t_*[\cdot]\vartheta)} (l(t_*[\cdot]\vartheta) \cdot z^{(2)}(t_*[\cdot]\vartheta)) \tag{2.10}$$

positive and nonincreasing with respect to β. The quantity $|l(t_*[\cdot]\vartheta)|_2^*$ is the norm adjoint to norm (2.8), i.e.,

$$|l(t_*[\cdot]\vartheta)|_2^* = \max\left\{\left(\int_{t_*}^{\vartheta} \|l[\tau]\|^2 d\tau\right)^{1/2}, \quad |l[\vartheta]|^*\right\} \tag{2.11}$$

where $|l|^*$ is the norm adjoint to the norm $|z|$ chosen in (2.8). The definitions of quantities σ and $\varkappa$ are well posed since all the extrema in (2.9) and (2.10) are, as a consequence of the above-mentioned properties of sets S, $W^{(1)}$ and $W^{(2)}$, actually reached on the corresponding sets of arguments.

Substituting the expression for $z^{(1)}(t_*[\cdot]\vartheta)$ into (2.10), using the Cauchy formula (2.7) for computing $z(t^*[\cdot]\vartheta)$, then changing the order of integration and introducing the functional

$$\psi(\tau, l(t_*[\cdot]\vartheta)) = \int_\tau^\vartheta l'[t]\, X(t, \tau)\, \mu(dt), \quad t^* \leqslant \tau \leqslant \vartheta \tag{2.12}$$

where the prime denotes transposition, we can prove that expression (2.10) takes the form

$$\begin{aligned} &\varkappa(z^*(t_*[\cdot]t^*), \beta, M, l(t_*[\cdot]\vartheta)) = \\ &(l(t_*[\cdot]\vartheta)\cdot z^*_{t_*}(t_*[\cdot]t^*)_\vartheta) + (l(t_*[\cdot]\vartheta)\cdot X_{t_*}(t^*[\cdot, t^*]\vartheta)_\vartheta z^*) + \\ &\int_{t^*}^{\vartheta} \max_{v\in Q}\min_{u\in P}[\psi(\tau, l(t_*[\cdot]\vartheta)) f_0(\tau, u, v)]\, d\tau - \\ &\max_{z^{(2)}(t_*[\cdot]\vartheta)} (l(t_*[\cdot]\vartheta)\cdot z^{(2)}(t_*[\cdot]\vartheta)) \end{aligned} \tag{2.13}$$

Here $Y_{t_*}(\tau_*[\cdot]\tau^*)_\vartheta$ is a function coinciding with function $Y(\tau_*[\cdot]\tau^*)$ on the interval $[\tau_*, \tau^*] \subset [t_*, \vartheta]$ and vanishing at the remaining points of interval $[t_*, \vartheta]$. When proving the transition from (2.10) to (2.13) it is important that the functions $\{\psi(\tau, l(t_*[\cdot]\vartheta)),\ |l(t_*[\cdot]\vartheta)|_2^* \leqslant 1\}$ be uniformly bounded and equicontinuous on the interval $[t^*, \vartheta]$ and if the sequence $\{l^{(k)}(t_*[\cdot]\vartheta),\ k = 1, 2, \ldots\}$ converges weakly in $L_\mu^{(2)}[t_*, \vartheta]$ to some function $l^*(t_*[\cdot]\vartheta)$, then the corresponding sequence $\{\psi(\tau, l^{(k)}(t_*[\cdot]\vartheta)),\ k = 1, 2, \ldots\}$ converges to function $\psi(\tau, l^*(t_*[\cdot]\vartheta))$ in $R^{(n-1)}$ uniformly with respect to τ.

Under a certain additional condition a reversion to problem (2.9), (2.13) enables us to find the value $\rho(t_*, x_*)$ of the differential game being analyzed. This additional regularity condition is formulated as follows. Suppose that $z^*(t_*[\cdot]t^*)$, β and M have been fixed and let $\sigma(z^*(t_*[\cdot]t^*), \beta, M) > 0$. Then for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $t^\circ : (t^\circ - t^*) \leqslant \delta(\varepsilon)$ and every action $v^*(t^*[\cdot]t^\circ)$ there exists an action $g^*(t^*[\cdot]t^\circ)$, consistent with action $v^*(t^*[\cdot]t^\circ)$, such that

$$\int_{t^*}^{t^\circ} \psi(\tau, l^\circ(t_*[\cdot]\vartheta))\, g^*[\tau]\, d\tau \leqslant \int_{t^*}^{t^\circ} \max_{v\in Q}\min_{u\in P}[\psi(\tau, l^\circ(t_*[\cdot]\vartheta)) f_0(\tau, u, v)]\, d\tau + \varepsilon(t^\circ - t^*) \tag{2.14}$$

for each element $l^\circ(t_*[\cdot]\vartheta)$ being a solution of problem (2.9), (2.13), corresponding to the chosen $z^*(t_*[\cdot]t^*)$, β and M. The regularity condition is automatically fulfilled if the maximum in (2.9) is reached on a single element $l^\circ(t_*[\cdot]\vartheta)$.

Let $\{t_*, z_*\}$ be some position, where $t_0 \leqslant t_* \leqslant \vartheta$, $z_* \in G^*[t_*]$. By the symbol $\rho^*(t_*, z_*)$ we denote the least upper bound of those numbers β for which $\sigma(z^*(t_*[\cdot]t^*), \beta, M) > 0$, where $z^*[t_*] = z_*$. It turns out that $\rho^*(t_*, z_*) = \rho_*(t_*, z_{1*}, \ldots, z_{n*}) + z_{n+1*}$, where $\rho_*(t_*, z_{1*}, \ldots, z_{n*}) = \rho^*(t_*, z_{1*}, \ldots, z_{n*}, 0)$. Let $\{t_*, x_*\}$ be an arbitrary position from domain G_0 and let the regularity condition be fulfilled; then the following assertion is valid: The value $\rho(t_*, x_*)$ of the game being examined is determined by the relation

$$\rho(t_*, x_*) = \rho_*(t_*, x_{1*}, \ldots, x_{n*}) = \rho^*(t_*, x_{1*}, \ldots, x_{n*}, 0) \tag{2.15}$$

The proof of this assertion is carried out by a scheme from the theory of auxiliary program constructions. Here, however, we need to take into account accurately the supplementary circumstances connected with the functional nature of the elements $l(t_*[\cdot]\vartheta)$. In this regard we emphasize that although the elements of the proposed program construction allowing us to compute the game's value, and, consequently, to construct the optimal strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$, bear a functional nature, the optimal control algorithms derived from these elements lead, in the final analysis, to the construction of forces as functions of a finite-dimensional description of the current position $\{t, x_1[t], \ldots, x_n[t]\}$ of the original object (2.1). However, if the integrand in the functional (1.1) being optimized is such that $\omega_x(t, x) \equiv 0$ in (2.2), the elements of the program construction being considered also are finite-dimensional in character. In this case problem (2.9), (2.13) is transformed to the following problem on the maximum of a function of the $(n+1)$st variable:

$$\sigma(t_*, z_*, \beta, M) = \max_{|l|^* \leqslant 1} \varkappa(t_*, z_*, \beta, M, l) \tag{2.16}$$

$$\varkappa(t_*, z_*, \beta, M, l) = \langle l \cdot X(\vartheta, t_*) z_* \rangle + \int_{t_*}^{\vartheta} \max_{v \in Q} \min_{u \in P} \langle l \cdot X(\vartheta, \tau) f_0(\tau, u, v) \rangle \, d\tau - \max_{z^{(2)} \in W^{(2)}(\beta, M)} \langle l \cdot z^{(2)} \rangle \tag{2.17}$$

where $|l|^*$ is the norm adjoint to the chosen norm $|z|$, while $W^{(2)}(\beta, M)$ is the set of those $z^{(2)} \in R^{n+1}$ for which $\varphi_0(z^{(2)}) \equiv \varphi(w) + w_{n+1} \leqslant \beta$ and $|z^{(2)}| \leqslant M$. In concluding this section we formulate an auxiliary statement which proves useful when solving concrete problems.

Let there be given fixed numbers β and $c > 0$ and a position set $G^\circ = \{\{t_* z_*\}, t_0 \leqslant t_* \leqslant \vartheta, z_* \in G^* [t_*]\}$. Let a number $M_* > 0$ exist such that the set $W^{(2)}(\beta, M_*)$ and $G_c = \{\{t_*, z_*\}: \sigma(t_*, z_*, \beta, M_*) < c\} \cap G^\circ$ is not empty, where $\sigma(t_*, z_*, \beta, M_*)$ is computed by formula (2.16) with $W^{(2)}(\beta, M_*)$. Then we can find a number $M^* > M_*$ satisfying the condition: if l° is the maximizing vector for $\varkappa(t_*, z_*, \beta, M^*, l)$ when $\sigma(t_*, z_*, \beta, M^*) < c$, then this vector l° remains maximizing also for $\varkappa(t_*, z_*, \beta, M, l)$ with $W^{(2)}(\beta, M)$ for all $M > M^*$, and, in this connection, $\sigma(t_*, z_*, \beta, M) = \sigma(t_*, z_*, \beta, M^*)$.

3. Let us illustrate the method for solving problem (2.16), (2.17) by an example when the Eqs. (2.1) of motion are

$$\frac{d^2 q}{dt^2} = u + v, \quad q = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad \|u\| \leqslant 2, \quad \|v\| \leqslant 1 \tag{3.1}$$

and function (1.1) is

$$\gamma = \int_{t_*}^{\vartheta} \langle u \cdot v \rangle \, dt + \|q[\vartheta]\|, \quad 0 = t_0 \leqslant t_* \leqslant \vartheta \tag{3.2}$$

We reduce system (3.1) to the normal form

$$x_1^{\cdot} = x_3, \quad x_3^{\cdot} = u_1 + v_1, \quad x_2^{\cdot} = x_4, \quad x_4^{\cdot} = u_2 + v_2 \tag{3.3}$$

Then

$$\gamma = \int_{t_*}^{\vartheta} \langle u \cdot v \rangle \, dt + (x_1^2[\vartheta] + x_2^2[\vartheta])^{1/2} \tag{3.4}$$

The model's Eqs. (2.3) and the functional (2.5) have the form

$$w_1^{\cdot} = w_3, \quad w_3^{\cdot} = u_1 + v_1, \quad w_2^{\cdot} = w_4, \quad w_4^{\cdot} = u_2 + v_2, \quad w_5^{\cdot} = u_1 v_1 + u_2 v_2 \tag{3.5}$$

$$\gamma_0(z(t_*[\cdot]\vartheta)) = w_5[\vartheta] + (w_1^2[\vartheta] + w_2^2[\vartheta])^{1/2} \tag{3.6}$$

Here it is convenient to choose the norm

$$|z| = \|w\| + |w_5| = (w_1^2 + \ldots + w_4^2)^{1/2} + |w_5|$$

for the vector $z = \{w_1, \ldots, w_4, w_5\}$. Then

$$|l|^* = \max\{(l_1^2 + \ldots + l_4^2)^{1/2}, |l_5|\}$$

In accord with (2.16) and (2.17), if the regularity condition (2.14) is fulfilled, then for determining the game's value $\rho(t_*, x_*)$ we obtain the problem

$$\sigma(t_*, z_*, \beta, M^*) = \max_{|l|^* \leqslant 1} \varkappa(t_*, z_*, \beta, M^*, l) \tag{3.7}$$

where the number M^* is chosen in accordance with the conditions of the auxiliary statement, $z_* = \{x_{1*}, \ldots, x_{4*}, 0\}$, while the expression for $\varkappa(t_*, z_*, \beta, M^*, l)$ (2.17) has the form

$$\begin{aligned} \varkappa(t_*, z_*, \beta, M^*, l) = {} & [l_1(x_{1*} + x_{3*} \cdot (\vartheta - t_*)) + \\ & l_2(x_{2*} + x_{4*} \cdot (\vartheta - t_*)) + l_3 x_{3*} + l_4 x_{4*}] + \int_{t_*}^{\vartheta} \max_{\|v\| \leqslant 1} \min_{\|u\| \leqslant 2} [l_1(u_1 + v_1)(\vartheta - \tau) + \\ & l_2(u_2 + v_2)(\vartheta - \tau) + l_3(u_1 + v_1) + \\ & l_4(u_2 + v_2) + l_5 \langle u \cdot v \rangle] \, d\tau - \max_{z^{(2)} \in W^{(2)}(\beta, M^*)} \langle l \cdot z^{(2)} \rangle \end{aligned} \tag{3.8}$$

By the definition of set $W^{(2)}(\beta, M^*)$, with due regard to (3.6) we have

$$W^{(2)}(\beta, M^*) = \{z^{(2)} : (w_1^2 + w_2^2)^{1/2} + w_5 \leqslant \beta, \quad |z^{(2)}| \leqslant M^*\} \tag{3.9}$$

From the form of $\varkappa(t_*, z_*, \beta, M^*, l)$ in (3.8) and $W^{(2)}(\beta, M^*)$ in (3.9), using the auxiliary statement, we can deduce that $l_3^\circ = l_4^\circ = 0$, $l_5^\circ = 1$, and then we obtain

$$\max_{z^{(2)} \in W^{(2)}(\beta, M^*)} \langle l^\circ \cdot z^{(2)} \rangle = \max_{(w_1^2 + w_2^2)^{1/2} + w_5 \leqslant \beta} (l_1^\circ w_1 + l_2^\circ w_2 + w_5) = \beta \tag{3.10}$$

Substituting (3.10) into (3.8), introducing the two-dimensional vector $s[\tau] = \{l_1\cdot(\vartheta-\tau),\ l_2\cdot(\vartheta-\tau)\}$, and allowing for (2.15), we find that

$$\rho(t_*, x_*) = \sup\{\beta : \sigma(t_*, z_*, \beta, M^*) > 0\} = \tag{3.11}$$
$$\max_{(l_1^2+l_2^2)\leqslant 1} \{[l_1(x_{1*} + x_{3*}\cdot(\vartheta - t_*)) + l_2(x_{2*} + x_{4*}\cdot(\vartheta - t_*))] +$$
$$\int_{t_*}^{\vartheta} \max_{\|v\|\leqslant 1} \min_{\|u\|\leqslant 2} [\langle s[\tau]\cdot u\rangle + \langle s[\tau]\cdot v\rangle + \langle u\cdot v\rangle]\, d\tau\}$$

when the regularity condition is fulfilled. It can be verified that the integrand in (3.11) equals $\lambda(\|s[\tau]\|)$, where

$$\lambda(\|s[\tau]\|) = \begin{cases} -\|s[\tau]\|^2, & \|s[\tau]\| \leqslant 1 \\ -3\|s[\tau]\| + 2, & 1 \leqslant \|s[\tau]\| \leqslant 2 \\ -\|s[\tau]\| - 2, & \|s[\tau]\| \geqslant 2 \end{cases} \tag{3.12}$$
$$\|s[\tau]\| = (\vartheta - \tau)\, r, \quad r = (l_1^2 + l_2^2)^{1/2}$$

Hence, the ratio l_2/l_1 does not for a nonvarying r influence the magnitude of the integral in (3.11). But then we can set

$$l_1 = r(x_{1*} + x_{3*}\cdot(\vartheta - t_*))/k, \quad l_2 = r(x_{2*} + x_{4*}\cdot(\vartheta - t_*))/k \tag{3.13}$$
$$k = [(x_{1*} + x_{3*}\cdot(\vartheta - t_*))^2 + (x_{2*} + x_{4*}\cdot(\vartheta - t_*))^2]^{1/2}$$

when $k \neq 0$; but when $k = 0$ we have $l_1^\circ = l_2^\circ = 0$, and with due regard to (3.13) problem (3.11) is reduced to the following problem on the maximum:

$$\rho(t_*, x_*) = \max_{0\leqslant r\leqslant 1} \left\{rk + \int_{t_*}^{\vartheta} \lambda(\|s[\tau]\|)\, d\tau\right\} \tag{3.14}$$

Computing in (3.14) the integral of the function $\lambda(\|s[\tau]\|)$ from (3.12) and denoting $T_* = \vartheta - t_*$, we finally find

$$\rho(t_*, x_*) = \max_{0\leqslant r\leqslant 1} \chi(T_*, x_*, r) \tag{3.15}$$
$$\chi(T_*, x_*, r) = rk - \begin{cases} r^2T_*^3/3; & r \in [0, T_*^{-1}] \cap [0, 1] \\ 3rT_*^2/2 + 5/(6r) - 2T_*; & r \in [T_*^{-1}, 2T_*^{-1}] \cap [0, 1] \\ rT_*^2/2 - 19/(6r) + 2T_*; & r \in [2T_*^{-1}, 1] \cap [0, 1] \end{cases}$$

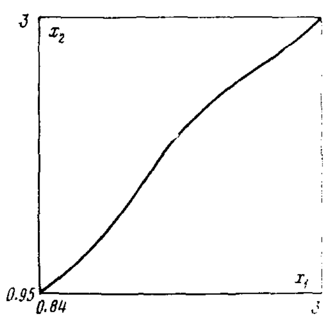

Fig.1

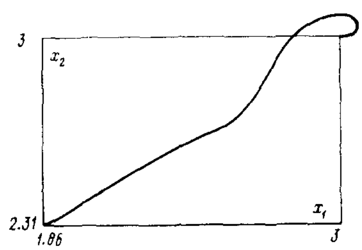

Fig.2

It remains to ascertain whether the regularity condition (2.14) holds. We can note that the function $\chi(\vartheta - t_*, x_*, r)$ in (3.15) for $\vartheta \leqslant \vartheta_* = 2$ is concave with respect to r for any position $\{t_*, x_*\} \in G_0$ and, consequently, the maximum in (3.15) is reached for the single value r°. Hence, the maximum over l in the original problem (3.7), (3.8) with $\sigma(t_*, z_*, \beta, M^*) > 0$ is reached on a single vector l°. In such a case, as was noted, the regularity condition is fulfilled. It can be shown that condition (2.14) holds also for $\vartheta_* < \vartheta \leqslant \vartheta_* + \eta$, where η is some positive number, in spite of the fact that the vector l° now will not be unique for all positions $\{t_*, x_*\} \in G_0$. As ϑ increases further the regularity conditions is perhaps violated.

The optimal strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$ are constructed in a known manner from the game's value $\rho(t, x)$ computed from condition (3.15) /2/. In typical situations the given game was simulated on a computer for the initial data $\vartheta = 2$, $t_* = 0$, $x_{1*} = x_{2*} = 3$, $x_{3*} = x_{4*} = 0$. If both players are guided by the optimal strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$, then object (3.3) moves in the plane $\{x_1, x_2\}$ along a straight line passing through the origin, and the index γ of (3.4) coincides with the game's value $\rho(t_*, x_*) = 1.578$. Fig.1 shows the motion of object (3.3) when $u = u^\circ(\cdot)$ and $v = \{\cos \pi t, \sin \pi t\}$. Here $\gamma = 1.025 < \rho(t_*, x_*)$; Fig.2 shows the motion of object (3.3) when $v = v^\circ(\cdot)$ and $u = \{2 \cos \pi t, 2 \sin \pi t\}$. Here $\gamma = 2.655 > \rho(t_*, x_*)$.

REFERENCES

1. SUBBOTIN A.I. and CHENTSOV A.G., Optimization of Security in Control Problems. Moscow,NAUKA, 1981.
2. KRASOVSKII A.N., KRASOVSKII N.N. and TRET'IAKOV V.E., Stochastic programmed design for a deterministic positional differential game. PMM Vol.45, No.4, 1981.
3. KRASOVSKII A.N., Differential game for a position functional. Dokl. Akad. Nauk SSSR, Vol. 253, No.6, 1980.
4. TARLINSKII S.I., On a linear differential encounter game. Dokl. Akad. Nauk SSSR, Vol.209, No.6, 1973.
5. KOLMOGOROV A.N. and FOMIN S.V., Elements of the Theory of Functions and Functional Analysis. Bethesda, MD, Graylock Press, 1961.
6. DUNFORD N. and SCHWARTZ J.T., Linear Operators. Pt. I: General Theory. New York, Interscience Publ., 1958.

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